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Noncommutative spectral geometry, dissipation and the origin of quantization

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Abstract. We present a physical interpretation of the doubling of the algebra, which is the basic ingredient of the noncommutative spectral geometry, developed by Connes and collaborators as an approach to unification. We discuss its connection to dissipation and to the gauge structure of the theory. We then argue, following 't Hooft's conjecture, that noncommutative spectral geometry classical construction carries implicit in its feature of the doubling of the algebra the seeds of quantization.

1. Noncommutative Spectral Geometry

Unification of all forces, including gravity, remains one of the open issues in theoretical physics. As one approaches Planckian energy scales, the assumption that physics can be described by the sum of the Einstein-Hilbert and Standard Model (SM) actions breaks down and one must consider quantum gravity effects. In an attempt to provide a basis for describing the quantum nature of space-time, which may lead to the unification of all forces, Connes and collaborators developed Noncommutative Spectral Geometry (NCSG) which combines notions of noncommutative geometries [1, 2] with spectral triples. Within NCSG, the SM of electroweak and strong interactions is seen [3] as a phenomenological model, which specifies the geometry of space-time so that the Maxwell-Dirac action functional leads to the SM action.

This unification model lives by construction at high energy scales, offering an appropriate framework to address early universe cosmology [4] - [11]. This is however beyond the scope of this presentation. In what follows we attempt instead, to shed some light on how some criticisms raised against NCSG approach, and in particular its application in early universe cosmology, can be answered. More precisely, we will discuss the physical meaning of the choice of the *almost* commutative geometry and its relation to quantization [12]. In our discussion we will

also consider the relation of the NCSG formalism with the gauge structure of the theory and with dissipation.

Let us first however review the main elements of NCSG. It is based on a two-sheeted space, made from the product of a four-dimensional smooth compact Riemannian manifold \mathcal{M} (a continuous geometry for space-time) with a fixed spin structure, by a discrete noncommutative space \mathcal{F} (an internal geometry for the SM) composed by only two points. The noncommutative nature of the discrete space \mathcal{F} is given by a spectral triple $(\mathcal{A}, \mathcal{H}, \mathcal{D})$, where \mathcal{A} is an involution of operators on the finite-dimensional Hilbert space \mathcal{H} of Euclidean fermions, and \mathcal{D} is a self-adjoint unbounded operator in \mathcal{H} . The space \mathcal{H} is the Hilbert space $L^2(\mathcal{M}, S)$ of square integrable spinors S on \mathcal{M} and the algebra \mathcal{A} is the algebra $\mathcal{A} = C^\infty(\mathcal{M})$ of smooth functions on \mathcal{M} and acts in \mathcal{H} by multiplication operators. The operator \mathcal{D} is the Dirac operator $\not{D}_{\mathcal{M}} = \sqrt{-1}\gamma^\mu \nabla_\mu^s$ on the spin Riemannian manifold \mathcal{M} .

Within NCSG all information about space is encoded in the algebra of coordinates \mathcal{A} . Assuming the algebra \mathcal{A} constructed in the geometry $\mathcal{M} \times \mathcal{F}$ is symplectic-unitary, it must be of the form [13]

$$\mathcal{A} = M_a(\mathbb{H}) \oplus M_k(\mathbb{C}) , \quad (1)$$

with $k = 2a$ and \mathbb{H} being the algebra of quaternions. The field of quaternions \mathbb{H} plays an important rôle in this construction and its choice remains to be explained. To obtain the SM one assumes quaternion linearity. The first possible value for the even number k is 2, corresponding to a Hilbert space of four fermions, but this choice is ruled out from the existence of quarks. The next possible value is $k = 4$ leading to the correct number of $k^2 = 16$ fermions in each of the three generations.

The noncommutative spectral geometry model is based upon the spectral action principle stating that, within the context of a product noncommutative geometry, the bare bosonic Euclidean action is given by the trace of the heat kernel associated with the square of the noncommutative Dirac operator and is of the form

$$\text{Tr}(f(D/\Lambda)) ; \quad (2)$$

f is a cut-off function, Λ fixes the energy scale, D and Λ have physical dimensions of a mass. This action can be seen *à la* Wilson as the bare action at the mass scale Λ . The fermionic term can be included in the action functional by adding $(1/2)\langle J\psi, D\psi \rangle$, where J is the real structure on the spectral triple and ψ is a spinor in the Hilbert space \mathcal{H} of the quarks and leptons.

For the four-dimensional Riemannian geometry, the trace $\text{Tr}(f(D/\Lambda))$ is expressed perturbatively in terms of the geometrical Seeley-deWitt coefficients a_n , which are known for any second order elliptic differential operator, as [14, 15, 16, 17]

$$\begin{aligned} \text{Tr}(f(D/\Lambda)) &\sim 2\Lambda^4 f_4 a_0 + 2\Lambda^2 f_2 a_2 + f_0 a_4 + \cdots \\ &+ \Lambda^{-2k} f_{-2k} a_{4+2k} + \cdots , \end{aligned} \quad (3)$$

where the smooth even cut-off function f , which decays fast at infinity, appears through its momenta f_k given by:

$$\begin{aligned} f_0 &\equiv f(0) \\ f_k &\equiv \int_0^\infty f(u) u^{k-1} du , \quad \text{for } k > 0 , \\ f_{-2k} &= (-1)^k \frac{k!}{(2k)!} f^{(2k)}(0) . \end{aligned}$$

Since its Taylor expansion at zero vanishes, the asymptotic expansion Eq. (3) reduces to

$$\text{Tr}(f(D/\Lambda)) \sim 2\Lambda^4 f_4 a_0 + 2\Lambda^2 f_2 a_2 + f_0 a_4 . \quad (4)$$

The cut-off function f plays a rôle only through its three momenta f_0, f_2, f_4 , which are three real parameters, related to the coupling constants at unification, the gravitational constant, and the cosmological constant, respectively. The term in Λ^4 gives a cosmological term, the term in Λ^2 gives the Einstein-Hilbert action functional, and the Λ -independent term yields the Yang-Mills action for the gauge fields corresponding to the internal degrees of freedom of the metric.

In this purely geometric approach to the SM, the fermions provide the Hilbert space of a spectral triple for the algebra, while the bosons are obtained through inner fluctuations of the Dirac operator of the product geometry. The computation of the asymptotic expression for the spectral action functional results to the full Lagrangian for the Standard Model minimally coupled to gravity, with neutrino mixing and Majorana mass terms.

Finally, let us clarify in which sense we talk of dissipation in what follows. This is necessary because the Standard Model, as is well known, is Quantum Field Theory (QFT) model describing a closed (nondissipative) system. Dissipation enters our discussion of the implications of the algebra doubling in the specific sense one observes that in electrodynamics neither the energy-momentum tensor of the matter field, nor that of the gauge field, are conserved. However, $\partial_\mu T_{\text{matter}}^{\mu\nu} = eF^{\mu\nu}j_\mu = -\partial_\mu T_{\text{gauge field}}^{\mu\nu}$. Thus, the *total* $T_{\text{total}}^{\mu\nu} = T_{\text{matter}}^{\mu\nu} + T_{\text{gauge field}}^{\mu\nu}$, which is the energy-momentum tensor of the *closed* system {matter field, electromagnetic field} is conserved: each element of the couple is *open* (dissipating) on the other one, although the *closeness* of the total system is ensured. In this sense, dissipation considerations in our discussion below do not spoil the closeness of the SM.

2. The algebra doubling and the gauge structure

In this Section we study the relation between the two-sheeted space in the NCSG construction and the gauge structure of the theory. One central ingredient in NCSG is indeed the “doubling” of the algebra $\mathcal{A} \rightarrow \mathcal{A}_1 \otimes \mathcal{A}_2$ acting on the “doubled” space $\mathcal{H} = \mathcal{H}_1 \otimes \mathcal{H}_2$. Such a doubling is the formal realization of the NCSG two-sheeted space. Let us first observe that the doubling of the algebra is also present in the standard Quantum Mechanics (QM) formalism of the density matrix and Wigner function. Indeed, the expression of the Wigner function is [18]

$$W(p, x, t) = \frac{1}{2\pi\hbar} \int \psi^* \left(x - \frac{1}{2}y, t \right) \psi \left(x + \frac{1}{2}y, t \right) e^{-i\frac{py}{\hbar}} dy .$$

By putting $x_\pm = x \pm \frac{1}{2}y$, the associated density matrix is

$$W(x_+, x_-, t) \equiv \langle x_+ | \rho(t) | x_- \rangle = \psi^*(x_-, t) \psi(x_+, t) , \quad (5)$$

The coordinate $x(t)$ of a quantum particle is thus split into two coordinates $x_+(t)$ (going forward in time) and $x_-(t)$ (going backward in time). The forward and the backward in time evolution of the density matrix $W(x_+, x_-, t)$ is then described by “two copies” of the Schrödinger equation:

$$i\hbar \frac{\partial \langle x_+ | \rho(t) | x_- \rangle}{\partial t} = H \langle x_+ | \rho(t) | x_- \rangle , \quad (6)$$

where H is given in terms of the two Hamiltonian operators H_\pm as $H = H_+ - H_-$. The connection with Alain Connes’ discussion of spectroscopic experiments and the algebra doubling is thus evident: the density matrix and the Wigner function *require* the introduction of a “doubled” set of coordinates (x_\pm, p_\pm) and of their respective algebras. Use of Eq. (6) shows immediately that the eigenvalues of H are directly the Bohr transition frequencies $h\nu_{nm} = E_n - E_m$, which was the first hint towards an explanation of spectroscopic structure.

The need to double the degrees of freedom is implicit even in the classical theory when considering the Brownian motion and it is related to dissipation. In the classical Brownian theory one has the equation of motion [19]

$$m\ddot{x}(t) + \gamma\dot{x}(t) = f(t) , \quad (7)$$

where $f(t)$ is a random (Gaussian distributed) force: $\langle f(t)f(t') \rangle_{\text{noise}} = 2\gamma k_B T \delta(t - t')$. By averaging over the fluctuating force f , one obtains [19]

$$\langle \delta[m\ddot{x} + \gamma\dot{x} - f] \rangle_{\text{noise}} = \int \mathcal{D}y \langle \exp[\frac{i}{\hbar} \int dt L_f(\dot{x}, \dot{y}, x, y)] \rangle_{\text{noise}} , \quad (8)$$

where

$$L_f(\dot{x}, \dot{y}, x, y) = m\dot{x}\dot{y} + \frac{\gamma}{2}(x\dot{y} - y\dot{x}) + fy . \quad (9)$$

Note that \hbar is introduced solely for dimensional reasons. We thus see that the constraint condition at the classical level introduced a new coordinate y , and the system equations are obtained:

$$m\ddot{x} + \gamma\dot{x} = f , \quad m\ddot{y} - \gamma\dot{y} = 0 . \quad (10)$$

The x -system is an *open* system. In order to set up the canonical formalism it is required to *close* the system; this is the rôle of the y -system, which is the time-reversed copy of the x -system. The $\{x - y\}$ system is thus a closed system. We also remark that the exact expression for the imaginary part of the action reads [20, 21]

$$\text{Im}\mathcal{S}[x, y] = \frac{1}{2\hbar} \int_{t_i}^{t_f} \int_{t_i}^{t_f} dt ds N(t - s) y(t) y(s) , \quad (11)$$

where $N(t - s)$ denotes the quantum noise in the fluctuating random force given by the Nyquist theorem [20]. From Eq. (11) we see that nonzero y yields an “unlikely process” in the classical limit “ $\hbar \rightarrow 0$ ”, in view of the large imaginary part of the action. At quantum level, instead, nonzero y may allow quantum noise effects arising from the imaginary part of the action [20]. We thus see that the second sheet cannot be neglected: in the perturbative approach one may drop higher order terms in the action functional expansion, since they correspond to unlikely processes at the classical level. However, these terms may be responsible for quantum noise corrections and thus, in order to not preclude the quantization effects, one should keep them.

Let us unveil now the relation between the two-sheeted space in the NCSG construction and the gauge structure of the theory. Consider the equation of the classical one-dimensional damped harmonic oscillator

$$m\ddot{x} + \gamma\dot{x} + kx = 0 , \quad (12)$$

with time independent m , γ and k , which is a simple prototype of open systems. In the canonical formalism for open systems, the doubling of the degrees of freedom is required in such a way as to complement the given open system with its time-reversed image, playing the rôle of the “bath”, thus obtaining a globally closed system for which the Lagrangian formalism is well defined. Thus we consider the oscillator in the *doubled* y coordinate

$$m\ddot{y} - \gamma\dot{y} + ky = 0 . \quad (13)$$

The system of the oscillators Eq. (12) and Eq. (13) is then a closed system described by the Lagrangian density Eq. (9) where we put $f = kx$. The canonically conjugate momenta p_x and p_y can now be introduced as customary. Let us use the coordinates $x_1(t)$ and $x_2(t)$: $x_1(t) = (x(t) + y(t))/\sqrt{2}$ and $x_2(t) = (x(t) - y(t))/\sqrt{2}$. The motion equations are

$$m\ddot{x}_1 + \gamma\dot{x}_2 + kx_1 = 0 , \quad m\ddot{x}_2 + \gamma\dot{x}_1 + kx_2 = 0 , \quad (14)$$

and $p_1 = m\dot{x}_1 + (1/2)\gamma x_2$; $p_2 = -m\dot{x}_2 - (1/2)\gamma x_1$. Following Refs. [22, 23, 24, 25] we can now put $B \equiv \gamma c/e$, $\epsilon_{ii} = 0$, $\epsilon_{12} = -\epsilon_{21} = 1$ and introduce the vector potential as

$$A_i = \frac{B}{2} \epsilon_{ij} x_j \quad (i, j = 1, 2) . \quad (15)$$

The Lagrangian can be written then in the familiar form

$$L = \frac{1}{2m}(m\dot{x}_1 + \frac{e_1}{c}A_1)^2 - \frac{1}{2m}(m\dot{x}_2 + \frac{e_2}{c}A_2)^2 - \frac{e^2}{2mc^2}(A_1^2 + A_2^2) - e\Phi, \quad (16)$$

which describes two particles with opposite charges $e_1 = -e_2 = e$ in the (oscillator) potential $\Phi \equiv (k/2e)(x_1^2 - x_2^2) \equiv \Phi_1 - \Phi_2$ with $\Phi_i \equiv (k/2/e)x_i^2$ and in the constant magnetic field \mathbf{B} defined as $\mathbf{B} = \nabla \times \mathbf{A} = -B\hat{\mathbf{z}}$. Remarkably, we have the Lorentzian-like (pseudoeuclidean) metric in Eq. (16). The “minus” sign, implied by the doubling of the degrees of freedom, is crucial in our derivation (and in the NCSG construction).

In conclusion, the doubled coordinate, e.g., x_2 acts as the gauge field component A_1 to which the x_1 coordinate is coupled, and *vice versa*. The energy dissipated by one of the two systems is gained by the other one and *vice versa*, in analogy to what happens in standard electrodynamics as observed at the end of Section I. The interpretation is recovered of the gauge field as the bath or reservoir in which the system is embedded [24, 25]. The gauge structure thus appears intrinsic to the doubling procedure.

Such a conclusion can be also reached in the case of a fermion field. For brevity we do not report here the discussion for the fermion case, which can be found in [24, 25, 12]. We only observe that, considering as an example the Lagrangian of the massless free Dirac field $L = -\bar{\psi}\gamma^\mu\partial_\mu\psi$, the field algebra is doubled by introducing the fermion tilde-field $\tilde{\psi}(x)$ and the Lagrangian is rewritten as

$$\hat{L} = L - \tilde{L} = -\bar{\psi}\gamma^\mu\partial_\mu\psi + \bar{\tilde{\psi}}\gamma^\mu\partial_\mu\tilde{\psi}. \quad (17)$$

The tilde-system is a “copy” (with the same spectrum and couplings) of the ψ -system. The Hamiltonian for the system is of the form $\hat{H} = H - \tilde{H}$. The key point is that the matrix elements of the Lagrangian Eq. (17) in a conveniently introduced space of states $\mathcal{H}_\theta \equiv \{|0(\theta)\rangle\}$, where $|0(\theta)\rangle$ denotes the ground state (see [24, 25, 21, 12]), as well as of a more general Lagrangian than the simple one presently considered, are invariant under the simultaneous local gauge transformations of ψ and $\tilde{\psi}$. The label θ in \mathcal{H}_θ denotes the angle of a Bogoliubov transformation (see below). The tilde term $\bar{\tilde{\psi}}\gamma^\mu\partial_\mu\tilde{\psi}$ transforms in such a way to compensate the local gauge transformation of the ψ kinematic term, i.e. $\bar{\tilde{\psi}}(x)\gamma^\mu\partial_\mu\tilde{\psi}(x) \rightarrow \bar{\tilde{\psi}}(x)\gamma^\mu\partial_\mu\tilde{\psi}(x) + g\partial^\mu\alpha(x)\tilde{j}_\mu(x)$. This suggests to introduce the field A'_μ by

$$gj^{\bar{\mu}}(x)A'_\mu(x) \cong \bar{\tilde{\psi}}(x)\gamma^{\bar{\mu}}\partial_\mu\tilde{\psi}(x), \quad \bar{\mu} = 0, 1, 2, 3. \quad (18)$$

where the bar over μ means no summation over repeated indices. The symbol \cong denotes equality among matrix elements in \mathcal{H}_θ . Thus we find that A'_μ transforms as $A'_\mu(x) \rightarrow A'_\mu(x) + \partial_\mu\alpha(x)$, and one may identify, in \mathcal{H}_θ , A'_μ with the conventional U(1) gauge vector field.

One can also show that the variations of the gauge field tensor $F'_{\mu\nu}$ have their source in the current \tilde{j}_μ , which suggests that the tilde field plays the rôle of the “bath” or “reservoir”. Such an interpretation in terms of a reservoir, may thus be extended also to the gauge field A'_μ , which indeed acts in a way to “compensate” the changes in the matter field configurations due to the local gauge freedom.

Finally, it can be shown that in the formalism of the algebra doubling a relevant rôle is played by the noncommutative q -deformed Hopf algebra [26], pointing to a deep physical meaning of the noncommutativity in this construction. Indeed, the map $\mathcal{A} \rightarrow \mathcal{A}_1 \otimes \mathcal{A}_2$ is just the Hopf coproduct map $\mathcal{A} \rightarrow \mathcal{A} \otimes \mathbb{1} + \mathbb{1} \otimes \mathcal{A} \equiv \mathcal{A}_1 \otimes \mathcal{A}_2$ which duplicates the algebra. The Bogoliubov transformation of “angle” θ relating the fields $\psi(\theta; x)$ and $\tilde{\psi}(\theta; x)$ to $\psi(x)$ and $\tilde{\psi}(x)$, is known to be obtained by convenient combinations of the *deformed* coproduct operation of the form

$\Delta a_q^\dagger = a_q^\dagger \otimes q^{1/2} + q^{-1/2} \otimes a_q^\dagger$, where $q \equiv q(\theta)$ is the deformation parameters and a_q^\dagger are the creation operators in the q -deformed Hopf algebra [26]. These deformed coproduct maps are noncommutative and the deformation parameter is related to the condensate content of $|0(\theta)\rangle$. It is interesting to observe that the q -derivative is a finite difference derivative [26], which has to be compared with the fact that in the NCSG construction the derivative in the discrete direction is a finite difference quotient.

A relevant point is that the deformation parameter *labels* the θ -representations $\{|0(\theta)\rangle\}$ and, for $\theta \neq \theta'$, $\{|0(\theta)\rangle\}$ and $\{|0(\theta')\rangle\}$ are unitarily inequivalent representations of the canonical (anti-)commutation rules. This is a characteristic feature of quantum field theory [21, 27]. Its physical meaning is that an order parameter exists, which assumes different θ -dependent values in each of the representations. In other words, the deformed Hopf algebra structure induces the *foliation* of the whole Hilbert space into physically inequivalent subspaces.

3. Dissipation and quantization

By discussing classical, deterministic models, 't Hooft has conjectured that, provided some specific energy conditions are met and some constraints are imposed, loss of information might lead to a quantum evolution [28, 29, 30]. In agreement with 't Hooft's conjecture, on the basis of the discussion in the previous Sections and following Refs. [31, 32], we propose [12] that the NCSG classical construction carries *implicit* in its feature of the doubling of the algebra the seeds of quantization.

We consider the classical damped harmonic x -oscillator described by Eq. (12) and its time-reversed image Eq. (13). The Casimir operator \mathcal{C} and the (second) $SU(1, 1)$ generator J_2 are [23]

$$\mathcal{C} = \frac{1}{4\Omega m} \left[(p_1^2 - p_2^2) + m^2 \Omega^2 (x_1^2 - x_2^2) \right], \quad J_2 = \frac{m}{2} \left[(\dot{x}_1 x_2 - \dot{x}_2 x_1) - \Gamma r^2 \right], \quad (19)$$

where \mathcal{C} is taken to be positive and

$$\Gamma = \frac{\gamma}{2m}, \quad \Omega = \sqrt{\frac{1}{m} \left(k - \frac{\gamma^2}{4m} \right)}, \quad \text{with } k > \frac{\gamma^2}{4m}.$$

The system's Hamiltonian can be written as [31, 32]

$$H = \sum_{i=1}^2 p_i f_i(q), \quad (20)$$

with $p_1 = \mathcal{C}$, $p_2 = J_2$, $f_1(q) = 2\Omega$, $f_2(q) = -2\Gamma$, $\{q_i, p_i\} = 1$ and the other Poisson brackets are vanishing. The Hamiltonian Eq. (20) belongs to the class of Hamiltonians considered by 't Hooft. There, the $f_i(q)$ are nonsingular functions of the canonical coordinates q_i and the equations for the q 's, namely $\dot{q}_i = \{q_i, H\} = f_i(q)$, are decoupled from the conjugate momenta p_i . A complete set of observables, called *beables*, then exists, which Poisson commute at all times. The meaning of this is that the system admits a deterministic description even when expressed in terms of operators acting on some functional space of states $|\psi\rangle$, such as the Hilbert space [29]. Such a description in terms of operators and Hilbert space, does not imply *per se* quantization of the system. Quantization is achieved only as a consequence of dissipation. The Hamiltonian is written as $H = H_I - H_{II}$, with

$$H_I = \frac{1}{2\Omega\mathcal{C}} (2\Omega\mathcal{C} - \Gamma J_2)^2, \quad H_{II} = \frac{\Gamma^2}{2\Omega\mathcal{C}} J_2^2 \quad (21)$$

and we impose the constraint $J_2|\psi\rangle = 0$, which defines physical states and guaranties that H is bounded from below. We can then write

$$H|\psi\rangle = H_I|\psi\rangle = 2\Omega\mathcal{C}|\psi\rangle = \left(\frac{1}{2m} p_r^2 + \frac{K}{2} r^2 \right) |\psi\rangle, \quad (22)$$

with $K \equiv m\Omega^2$. We thus realize that H_I reduces to the Hamiltonian for the two-dimensional “isotropic” (or “radial”) harmonic oscillator $\ddot{r} + \Omega^2 r = 0$.

The physical states are invariant under time-reversal ($|\psi(t)\rangle = |\psi(-t)\rangle$) and periodical with period $\tau = 2\pi/\Omega$. Note that $H_I = 2\Omega\mathcal{C}$ has the spectrum $\mathcal{H}_I^n = \hbar\Omega n$, $n = 0, \pm 1, \pm 2, \dots$; since our choice has been that \mathcal{C} is positive, only positive values of n will be considered.

By exploiting the periodicity of the physical states $|\psi\rangle$ and following Ref. [33], one obtains

$$\frac{\langle\psi(\tau)|H|\psi(\tau)\rangle}{\hbar}\tau - \phi = 2\pi n \quad , \quad n = 0, 1, 2, \dots \quad .$$

Using $\tau = 2\pi/\Omega$ and $\phi = \alpha\pi$, where α is a real constant, leads to

$$\mathcal{H}_{I,\text{eff}}^n \equiv \langle\psi_n(\tau)|H|\psi_n(\tau)\rangle = \hbar\Omega \left(n + \frac{\alpha}{2} \right) \quad . \quad (23)$$

The index n signals the n dependence of the state and the corresponding energy. $\mathcal{H}_{I,\text{eff}}^n$ gives the effective n th energy level of the system, namely the energy given by \mathcal{H}_I^n corrected by its interaction with the environment. We conclude that the dissipation term J_2 of the Hamiltonian is responsible for the zero point ($n = 0$) energy: $E_0 = (\hbar/2)\Omega\alpha$. In QM the zero point energy is formally due to the nonzero commutator of the canonically conjugate q and p operators: the zero point energy is the “signature” of quantization. Our discussion thus shows that dissipation manifests itself as “quantization”. In other words, the (zero point) “quantum contribution” E_0 to the spectrum of physical states signals the underlying dissipative dynamics.

Consider the defining relation for temperature in thermodynamics (with $k_B = 1$): $\partial S/\partial U = 1/T$. Using $S \equiv (2J_2/\hbar)$ and $U \equiv 2\Omega\mathcal{C}$, Eq. (20) gives $T = \hbar\Gamma$. Provided S is identified with the entropy, $\hbar\Gamma$ can be regarded as the temperature. Thus, the “full Hamiltonian” Eq. (20) plays the rôle of the free energy \mathcal{F} , and $2\Gamma J_2$ represents the heat contribution in H (or \mathcal{F}). The fact that $2J_2/\hbar$ behaves as the entropy is not surprising since it controls the dissipative (thus irreversible loss of information) part of the dynamics.

The thermodynamical picture outlined above is also consistent with the results on the canonical quantization of open systems in quantum field theory [34].

4. The dissipative interference phase

Dissipation implies the appearance of a “dissipative interference phase” [35]. This provides a relation between dissipation and noncommutative geometry in the plane of the doubled coordinates and thus with the NCSG construction.

In the (x_+, x_-) plane (cf. Section 2), the components of forward and backward in time velocity $v_{\pm} = \dot{x}_{\pm}$ are given by

$$v_{\pm} = \frac{\partial H}{\partial p_{\pm}} = \pm \frac{1}{m} \left(p_{\pm} \mp \frac{\gamma}{2} x_{\mp} \right) \quad , \quad (24)$$

and they do not commute

$$[v_+, v_-] = i\hbar \frac{\gamma}{m^2} \quad . \quad (25)$$

It is thus impossible to fix these velocities v_+ and v_- as being identical [35]. By putting $mv_{\pm} = \hbar K_{\pm}$, a canonical set of conjugate position coordinates (ξ_+, ξ_-) may be defined by $\xi_{\pm} = \mp L^2 K_{\mp}$ so that

$$[\xi_+, \xi_-] = iL^2 \quad . \quad (26)$$

Equation (26) characterizes the noncommutative geometry in the plane (x_+, x_-) . In full generality, one can show [35] that an Aharonov–Bohm-type phase interference can always be associated with the noncommutative (X, Y) plane where

$$[X, Y] = iL^2 \quad ; \quad (27)$$

L denotes the geometric length scale in the plane. Consider a particle moving in the plane along two paths, \mathcal{P}_1 and \mathcal{P}_2 , starting and finishing at the same point, in a forward and in a backward direction, respectively. Let \mathcal{A} denote the resulting area enclosed by the paths. The phase interference ϑ may be shown [35] to be given by

$$\vartheta = \frac{\mathcal{A}}{L^2} . \quad (28)$$

and Eq. (27) in the noncommutative plane can be written as

$$[X, P_X] = i\hbar \quad \text{where} \quad P_X = \left(\frac{\hbar Y}{L^2} \right) . \quad (29)$$

The quantum phase interference between two alternative paths in the plane is thus determined by the noncommutative length scale L and the enclosed area \mathcal{A} .

Notice that the existence of a phase interference is connected to the zero point fluctuations in the coordinates; indeed Eq. (27) implies a zero point uncertainty relation $(\Delta X)(\Delta Y) \geq L^2/2$.

In the dissipative case, $L^2 = \hbar/\gamma$, we conclude that the quantum dissipative phase interference $\vartheta = \mathcal{A}/L^2 = \mathcal{A}\gamma/\hbar$ is associated with the two paths \mathcal{P}_1 and \mathcal{P}_2 in the noncommutative plane, provided $x_+ \neq x_-$.

5. Conclusions

We have shown that the central ingredient in the NCSG, namely the doubling of the algebra $\mathcal{A} = \mathcal{A}_1 \otimes \mathcal{A}_2$ acting on the space $\mathcal{H} = \mathcal{H}_1 \otimes \mathcal{H}_2$ is related to dissipation and to the gauge field structure. Thus the two-sheeted geometry must be considered to be the construction leading to gauge fields, required to explain the Standard Model. By exploiting 't Hooft's conjecture, according which loss of information within the framework of completely deterministic dynamics, might lead to a quantum evolution, we have argued that dissipation, implied by the algebra doubling, may lead to quantum features. We thus suggest that the NCSG classical construction carries *implicit* in the doubling of the algebra the seeds of quantization.

We have shown that in Alain Connes' two-sheeted construction, the doubled degree of freedom is associated with "unlikely processes" in the classical limit, and it may thus be dropped in higher order terms in the perturbative expansion. However, since these higher order terms are the ones responsible for quantum corrections, the second sheet cannot be neglected if one does not want to preclude quantization effects. In other words, the second sheet cannot be neglected once the universe entered the radiation dominated era. However, at the Grand Unified Theories scale, when inflation took place, the effect of gauge fields is fairly shielded.

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